

A remark on spaces of flat metrics with cone singularities of constant sign curvatures

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Abstract

By a result of W. P. Thurston, the moduli space of flat metrics on the sphere with prescribed n cone singularities of positive curvature is a complex hyperbolic orbifold of dimension $n-3$. The Hermitian form comes from the area of the metric. Using geometry of Euclidean polyhedra, we observe that this space has a natural decomposition into real hyperbolic convex polyhedra of dimensions $n-3$ and $\leq \frac{1}{2}(n-1)$.

By a result of W. Veech, the moduli space of flat metrics on a compact surface with prescribed cone singularities of negative curvature has a foliation whose leaves have a local structure of complex pseudo-spheres, coming again from the area of the metric. The form can be degenerate; its signature depends on the collection of angles. Using polyhedral surfaces in Minkowski space, we show that this moduli space has a natural decomposition into *spherical* convex polyhedra.

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1 Flat metrics

Let S be a compact oriented surface of genus g . Let us consider a gluing of Euclidean (non-degenerate) triangles along isometric edges, such that the resulting gluing is homeomorphic to S . Two gluings are equivalent if they differ by a finite number of *flips*, see Figure 2. We refer to [Tro07, Bon09] for more details.

An equivalence class of gluings defines a distance on S , which is isometric to the distance given by a flat Riemannian metric on S less some of the vertices of the triangles. Such a metric is a *flat metric* (with conical singularities) on S .

At a vertex p_i , if the sum α_i (the *cone angle*) of the angles of the triangles around p_i is different from 2π , then p_i is a *cone singularity*. The (*singular*) *curvature* at p_i is

$$k_i = 2\pi - \alpha_i .$$

See Figure 1 for examples.

In all the paper, n is the number of cone singularities.

Lemma 1.1 (Gauss–Bonnet formula).

$$\sum_{i=1}^n k_i = 2\pi(2 - 2g) . \quad (1)$$

Proof. Let m be a flat metric on S , with n cone singularities, T the number of triangles and E the number of edges. If $\chi(S)$ is the Euler characteristic of S , the Euler formula

$$T - E + n = 2 - 2g$$

gives, as for a triangulation $E = \frac{3}{2}T$,

$$2n - T = 2(2 - 2g) .$$

On the other hand, the sum of all the angles of all the triangles is equal to the sum of all the cone angles, hence

$$\pi T = \sum_{i=1}^n \alpha_i = 2\pi n - \sum_{i=1}^n k_i .$$

□

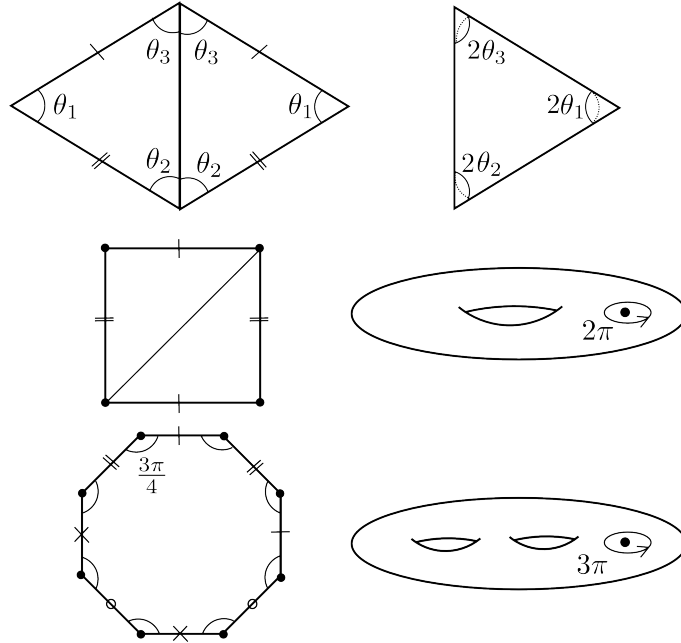


Figure 1: Examples of flat metrics for surfaces of different genus.

Let

$$\alpha = (\alpha_1, \dots, \alpha_n)$$

with α_i positive numbers. We denote

$$\mathcal{M}(\alpha) = \{\text{flat metrics on } S \text{ with cone angles } \alpha\} / \text{orientation-preserving similarities}$$

By a theorem of M. Troyanov, $\mathcal{M}(\alpha)$ is not empty if and only if α satisfies Gauss–Bonnet formula, and it has a structure of a complex manifold of (complex) dimension $3g - 3 + n$ [Tro86, Tro07].

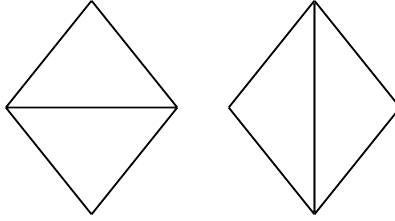


Figure 2: Flip.

1.1 Positive curvatures

Let us denote $(\alpha_1, \dots, \alpha_n)$ by α^+ if, for all i , $\alpha_i < 2\pi$ (i.e. all the curvatures are positive), and the $k_i = 2\pi - \alpha_i$ satisfy the Gauss–Bonnet formula. In particular, a metric m in $\mathcal{M}(\alpha^+)$ is necessarily a metric on the sphere. Moreover, as $\alpha_i > 0$, then $k_i = 2\pi - \alpha_i < 2\pi$. Hence by Gauss–Bonnet formula, in the case the k_i are all positive, $n \geq 3$.

Let us recall a celebrated construction of Thurston [Thu98, Sch15]. Let us cut m along a geodesic path connecting the cone points, and isometrically immerse the resulting flat disc onto the complex plane as a polygon (with possible self-intersections). If the flat metric comes from the induced metric on the boundary of a convex polytope of \mathbb{R}^3 , this is the same as considering the net of the polytope.

By Euler formula, the disc has $2(n-1)$ sides. Each side gives a vector of the plane, identified with a complex number. One can prove that it suffices to know $n-2$ of those complex parameters to recover the polygon, hence the metric. Up to similarities (i.e. multiplication by a non-zero complex number), this provides local charts from $\mathcal{M}(\alpha^+)$ to \mathbb{CP}^{n-3} . We will call the corresponding set in \mathbb{C}^{n-2} an *unfolding chart*.

The *area* of m is the sum of the area of the triangles. In an unfolding chart, the area extends to a Hermitian form on \mathbb{C}^{n-2} . By induction on n , using a procedure of chasing the curvature, Thurston proved that this form has signature $(1, n-3)$. As the image of the unfolding chart is in the positive cone of the area, $\mathcal{M}(\alpha^+)$ has a complex hyperbolic structure of dimension

$$n-3.$$

On the other hand, a famous theorem of A.D. Alexandrov says that there is a convex polytope P in the Euclidean space \mathbb{R}^3 such that the metric on the boundary of P is equal to m in $\mathcal{M}(\alpha^+)$ — up to triangulate the faces of P . Moreover, P is unique up to composition by Euclidean space isometries. There are two possibilities for P .

- Either P has empty interior. Hence it is the doubling (as in the first picture of Figure 1) of a convex polygon of the Euclidean plane \mathbb{R}^2 , that we also denote by P by abuse of notation. Let $H(P)$ be the set of convex polygons with edges parallel to the ones of P , up to translations. $H(P)$ naturally identifies with a convex cone in \mathbb{R}^{n-2} . Let $H_1(P)$ be the set of elements of $H(P)$ with area one. The area of the polygons extends to a quadratic form on \mathbb{R}^{n-2} . As it was noted by C. Bavard and É. Ghys [BG92], this quadratic form has signature $(1, n-3)$. In particular, $H_1(P)$ is a convex hyperbolic polyhedron of dimension

$$n-3.$$

See section 2.

- Otherwise, let us denote by $H(P)$ the set of convex polytopes with faces parallel to P , and having the same combinatorics as P , up to translations. $H(P)$ has a natural structure of a convex cone (with non empty interior) in \mathbb{R}^d . The dimension d is maximal when P is *simple*, i.e. it has exactly three edges at each vertex. In this case, $d = f-3$, where f is the number of faces of P . By Euler formula, if n is the number of vertices, $f = n/2 + 2$, and $d = n/2 - 1$ — the maximal dimension can be reached only if n is even. Also, the dimension d is minimal when there are only homothetic of P in $H(P)$ — hence $d = 1$.

Let $H_1(P)$ be the set of elements of $H(P)$ with area one. The area of the boundary of elements of $H(P)$ extends to a quadratic form on \mathbb{R}^d . As it was noted in [FI17], this quadratic form has signature $(1, d - 1)$. In particular, $H_1(P)$ is a convex hyperbolic polyhedron of dimension $d - 1$, i.e. between 0 and

$$n/2 - 2 .$$

See Section 3.

In any case, there is a linear injective map from \mathbb{R}^d to \mathbb{C}^{n-2} which map $H(P)$ into an unfolding chart of $\mathcal{M}(\alpha^+)$, and which is an isometry if both spaces are endowed with the area form. Heuristically, when a flat metric m comes from an element of $H(P)$, after unfolding in the plane, the parameters of $H(P)$ allow to move only the length of the edges. There are more freedom to deform m in the unfolding chart.

1.2 Negative curvatures

Let us denote $(\alpha_1, \dots, \alpha_n)$ by α^- if, for all i , $\alpha_i > 2\pi$ (i.e. all the curvatures are negative) and the $k_i = 2\pi - \alpha_i$ satisfy the Gauss–Bonnet formula. A metric m in $\mathcal{M}(\alpha^-)$ is necessarily a metric on a (compact) surface of genus $g > 1$. As far as we know, there is no particular construction for this case. But there is a general construction for any space of flat metrics with prescribed cone-angles due to W. Veech [Vee93, Ngu10, GP16], generalizing the one of Thurston.

A flat metric m in $\mathcal{M}(\alpha^-)$ can also be cut along a geodesic path connecting the cone points. Cutting along this path gives a flat metric on the disc, so there is only one face, and Euler formula gives that the number of edges of the disc is $2(2g - 1 + n)$. If the disc is isometrically immersed in the complex plane, one can show that k complex parameters suffice to recover the disc and hence the metric m , where

- $k = 2g - 2 + n$ if there is a cone angle which is not an entire multiple of 2π (case 1),
- $k = 2g - 1 + n$ if all cone angles are entire multiple of 2π (case 2).

This provides local charts into \mathbb{CP}^{k-1} of leaves of a foliation of $\mathcal{M}(\alpha^-)$, defined by prescribing the linear part of the holonomy of the flat metrics, into \mathbb{CP}^{k-1} . We call such leaves *Veech leaves* and the corresponding set in \mathbb{C}^k ant *unfolding charts*. In particular, the Veech leaves of $\mathcal{M}(\alpha^-)$ have complex dimension

$$2g + n - 3 \text{ or } 2g + n - 2 .$$

In an unfolding chart, the area form still extends to a Hermitian form on \mathbb{C}^k . Its signature is given by Theorem 14.6 in [Vee93]. Here *the signature depends on the cone angles*. In particular,

- In the case 2, the signature is $(g - 1, g - 1)$.
- If none of the cone angle is an integer multiple of 2π , there exists an integer ϵ (depending on the cone angles) such that the signature is

$$(g - 2 + n - \epsilon, g + \epsilon) .$$

See Table 1 for some examples with negative curvature in genus 2.

Also, an Alexandrov-like theorem says that there is a convex polyhedron P in the Minkowski space $\mathbb{R}^{2,1}$ invariant under the action of a cocompact lattice Γ of $SO(1, 2)$ such that the metric on the boundary of P/Γ is equal to a prescribed m in $\mathcal{M}(\alpha^-)$. Moreover, P is unique up to composition by Lorentzian linear isometries, [Fil11b, Bru]. Let us call P a Γ convex polyhedron, see section 5 for precise definitions. The following basic properties of Γ convex polyhedra will change dimensional computations in regard of the convex compact case:

- the set of Γ convex polyhedron is not stable under translations,
- they have non-empty interior.

(a_1, \dots, a_n)	signature	dimension
$(1, 1)$	$(1, 1)$	5
$(1/2, 1/2, 1/2, 1/2)$	$(3, 3)$	6
$(1/3, 1/3, 1/3, 1/3, 1/3, 1/3)$	$(5, 3)$	8
$(4/3, 1/3, 1/3)$	$(3, 2)$	5
$(1, 1/2, 1/2)$	$(2, 2)$	5
$(1, 1/3, 1/3, 1/3)$	$(3, 2)$	6

Table 1: Some example of the signature of the area form in genus 2 in the complex vector space containing an unfolding chart of $\mathcal{M}(\alpha^-)$ (here $\alpha_i = 2\pi(1 + a_i)$).

Let us denote by $H^\Gamma(P)$ the set of Γ convex polyhedra with faces parallel to P , and having the same combinatorics as P . $H^\Gamma(P)$ has a natural structure of a convex cone (with non empty interior) in \mathbb{R}^d . The dimension d is maximal when P is simple. In this case, $d = f$, where f is the number of faces of P in a fundamental domain for the action of Γ . By Euler formula, if n is the number of vertices, $f = n/2 + 2 - 2g$ — the maximal dimension can be reached only if n is even. Also, the dimension d is minimal when there are only homothetic of P in $H^\Gamma(P)$ — hence $d = 1$.

The function area from $H^\Gamma(P)$ to \mathbb{R}^+ gives the area (for the ambient Lorentzian metric) on the quotient by Γ of the boundary of the elements of $H^\Gamma(P)$. Let $H_1^\Gamma(P)$ be the set of elements of $H^\Gamma(P)$ with area equal to one. The area extends to a quadratic form on \mathbb{R}^d . Here, the area is *positive definite*, and $H_1^\Gamma(P)$ is a convex *spherical* polyhedron. Its dimension $d - 1$ is between 0 and

$$n/2 + 1 - 2g .$$

See Section 5.

There is a linear injective map from \mathbb{R}^d to \mathbb{C}^k which maps $H^\Gamma(P)$ into an unfolding chart of a Veech leaf of $\mathcal{M}(\alpha^-)$, and which is an isometry if both spaces are endowed with the area form.

2 Spaces of polygons

General references for mixed-volume and mixed-area are [Sch14, Ale05, Ale96]

Let P be a convex polygon (with non empty interior) on the plane, with n edges. Let us denote by

$$\tilde{H}(P) = \{\text{convex polygons with edges parallel to } P\}$$

$$H(P) = \tilde{H}(P)/\text{translations}$$

$$H_1(P) = H(P)/\text{homotheties} .$$

Let $h_i(P)$ be the support number of its i th edge: it is the distance from the origin to the line containing the edge. It is a signed distance: it is positive if 0 is in the same side of the line than P . Let $\ell_i(P)$ be the length of that edge. If this edge is bounded by the intersection of the lines with support numbers h_j and h_k is (see Figure 3) :

$$\ell_i(P) = \frac{1}{\sin \theta_j} (h_j + h_i \cos \theta_j) + \frac{1}{\sin \theta_k} (h_k + h_i \cos \theta_k) . \quad (2)$$

Then the area (=Lebesgue measure) of P is, as a sum of area of triangles:

$$\text{area}(P) = \frac{1}{2} \sum_{i=1}^n h_i(P) \ell_i(P) .$$

The *mixed area* is

$$\text{area}(P, Q) = \frac{1}{4} (\sum h_i(P) \ell_i(Q) + h_i(Q) \ell_i(P)) . \quad (3)$$

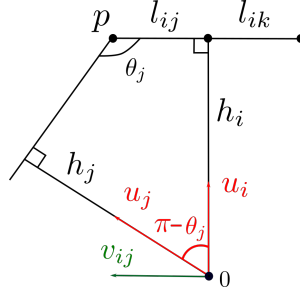


Figure 3: Length $\ell_i(P)$: $p = l_{ij}v_{ij} + h_i u_i$, $u_j = \cos(\pi - \theta_j)u_i + \sin(\pi - \theta_j)v_{ij}$. As $h_j = \langle u_j, p \rangle$, we obtain $l_{ij} = \frac{1}{\sin \theta_j} (h_j + h_i \cos \theta_j)$ and $l_i = l_{ij} + l_{ik}$.

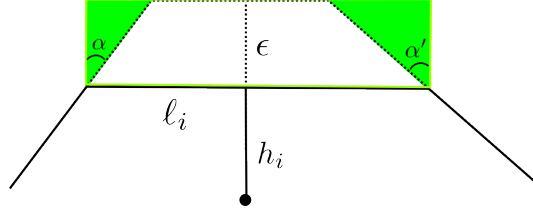


Figure 4: $\partial_i \text{area}(P) = \ell_i$. The area of the yellow rectangle minus the areas of the green triangles is $\epsilon \ell_i - \frac{1}{2} \epsilon^2 (\tan \alpha + \tan \alpha')$.

Lemma 2.1.

$$\text{area}(P, Q) = \frac{1}{2} \sum_{i=1}^n h_i(P) \ell_i(Q)$$

Proof. The fundamental remark is that $\partial_i \text{area}(P) \partial h_i = \ell_i(P)$, where ∂_i means that the variable is the i th support number, see Figure 4. So

$$\text{area}(P, Q) = \frac{1}{2} D_P \text{area}(Q) = \frac{1}{2} \sum h_i(Q) \partial_i \text{area}(P) = \frac{1}{2} \sum h_i(Q) \ell_i(P) .$$

Comparing with (3), one gets $\sum h_i(Q) \ell_i(P) = \sum h_i(P) \ell_i(Q)$. See [Fil11a] for another argument. \square

As elements of $\tilde{H}(P)$ are determined by their support numbers, we will identify $\tilde{H}(P)$ with a subset of \mathbb{R}^n . The map ℓ_i defined by (2) extends to a linear form on \mathbb{R}^n . The condition for $h \in \mathbb{R}^n$ to be in $\tilde{H}(P)$ is to have positive edges lengths. Hence, $\tilde{H}(P)$ is an open convex polyhedral cone in \mathbb{R}^n .

Moreover area extends to a symmetric bilinear form area_P on \mathbb{R}^n :

$$\text{area}_P(h, k) = \frac{1}{2} \sum_{i=1}^n h_i \ell_i(k) .$$

For $h, k \in \tilde{H}(P)$ representing convex polygons Q_1 and Q_2 , $\text{area}_P(h, k) = \text{area}(Q_1, Q_2)$.

For a point $x \in \mathbb{R}^2$, let h_x be the element of \mathbb{R}^n with entries $h_i = \langle x, n_i \rangle$. It may be seen as a support vector of $\{x\}$, with $\{x\}$ considered as a degenerated convex polygon in $\tilde{H}(P)$.

What can we say about the signature of area_P ?

1. **There is at least one positive direction.** This is only because for any $h \in \tilde{H}(P)$, $\text{area}_P(h) > 0$.
2. **The dimension of its kernel is at least two.** Indeed, let $x \in \mathbb{R}^2$. Then $Q + \{x\}$ is the translation of Q by the vector x , and

$$0 = \text{area}(Q + \{x\}) - \text{area}(Q) = 2 \text{area}(Q, \{x\}) ,$$

so for any $h \in \tilde{H}(P)$, $\text{area}_P(h, h_x) = 0$, and as $\tilde{H}(P)$ spans \mathbb{R}^n , $h_x \in \text{Ker}(\text{area}_P)$.

In order to be able to say more, we need to consider the bigger set \mathcal{K}_0^2 of plane convex bodies (i.e. compact convex sets with non empty interior). This set is stable under *Minkowski addition*

$$K_1 + K_2 = \{x + y | x \in K_1, y \in K_2\}$$

and (positive) homotheties: for $\lambda > 0$,

$$\lambda K = \{\lambda x | x \in K\} .$$

For elements of $\tilde{H}(P) \subset \mathbb{R}^n$, those operations corresponds to the ones of the vector space structure of \mathbb{R}^n . In particular, the mixed-area is a polarization of the area, i.e. it is “bilinear” for those operations.

Theorem 2.2 (Minkowski inequality). *For any $K_1, K_2 \in \mathcal{K}_0^2$,*

$$\text{area}(K_1, K_2)^2 \geq \text{area}(K_1)\text{area}(K_2) .$$

From this inequality, we can deduce the following information about area_P .

3. The positive index of area_P is one. Indeed, Minkowski inequality says that, for $h, k \in \tilde{H}(P)$,

$$\text{area}_P(h, k)^2 \geq \text{area}_P(h)\text{area}_P(k) .$$

Suppose that the positive index of area_P is larger than one. Then, for $h \in \tilde{H}(P)$, there exists $v \in \mathbb{R}^n$ orthogonal to h and $\text{area}_P(v, v) > 0$. But as $\tilde{H}(P)$ is open, there exists $\epsilon > 0$ such that $h + \epsilon v \in \tilde{H}(P)$. So area_P is positive definite on the plane spanned by v and h . But then Cauchy-Schwarz inequality applies, and contradicts Minkowski inequality.

Also, we need the description of the equality case in Minkowski inequality.

Theorem 2.3 (Equality case). *Equality occurs in Minkowski inequality if and only if there is $x \in \mathbb{R}^2$ and $\lambda > 0$ with $K_1 = \{x\} + \lambda K_2$.*

4. The dimension of the kernel of area_P is 2. Indeed, the description of the equality case in Minkowski inequality says that for $h, k \in \tilde{H}(P)$, $\text{area}_P(h, k) = \text{area}_P(h)\text{area}_P(k)$ if and only if there is $x \in \mathbb{R}^2$ and $\lambda > 0$ with $h = h_x + \lambda k$.

Let v belongs to the kernel. There exist $h, k \in \tilde{H}(P)$ such that $v = h - k$. From $\text{area}_P(h, v) = 0$ we deduce that $\text{area}_P(h, k) = \text{area}_P(h)$, and from $\text{area}_P(k, v) = 0$, that $\text{area}_P(h, k) = \text{area}_P(k)$. We then obtain equality in Minkowski inequality, and there is $x \in \mathbb{R}^2$ with $h = h_x + \lambda k$ and $v = h_x + (\lambda - 1)k$. From $\text{area}_P(v, v) = 0$ we deduce that $\lambda = 1$ so $v = h_x$.

Corollary 2.4. *The bilinear form area_P has signature $(1, 2, n - 3)$ on \mathbb{R}^n .*

In particular, $H_1(P)$ is a convex polyhedron in \mathbb{H}^{n-3} , which isometrically embeds into $\mathcal{M}(\alpha^+)$.

Proof. Let $Q \in \tilde{H}(P)$. Consider another copy of Q , glued to Q along an edge. This corresponds to the image of the metric on the sphere given by the doubling of Q for an unfolding chart. The parameters are the complex numbers given by the edges seen as vectors. In $\tilde{H}(P)$, the angles between the edges are fixed, and one can move the length of the edges, that are linear parameters with respect to the support vectors. \square

Remark 2.5. It is possible to have a lot of informations about $H_1(P)$ from P . First, $H_1(P)$ is a simple convex polyhedron. It has finite volume, and is compact if and only if P has no parallel edges. Actually $H_1(P)$ is an orthoscheme (roughly speaking, each facet meets non-orthogonally at most two other facets), and its dihedral angles can be easily computed from the angles of P . In particular one can find the Coxeter ones. Moreover, if P' has same angles than P , in the same order, except for two consecutive ones, then $H_1(P)$ and $H_1(P')$ can be glued along isometric facets. Performing a gluing for all permutations of angles, one obtains a hyperbolic cone-manifold, which embeds isometrically into $\mathcal{M}_1(\alpha^+)$, see [BG92, Fil11a] for more details.

All those properties do *not* hold for the polyhedra considered in the next section.

Remark 2.6. We deduced that spaces of convex polygons with the area form are convex hyperbolic polyhedra from the Minkowski inequality. But this inequality holds for any convex body of the plane. One can then deduce in a similar way that the set of plane convex bodies, up to translations and homotheties, is a convex subset of the infinite dimensional hyperbolic space. This is the subject of the last chapter of C. Debin PhD thesis [Deb16].

3 Area form on convex polytopes

Let P be a convex polytope (with non empty interior) of \mathbb{R}^3 , with f (codimension 1) faces.

$$\tilde{H}(P) = \{\text{convex polytope with faces parallel and same combinatorics than } P\}$$

$$H(P) = \tilde{H}(P)/\text{translations}$$

$$H_1(P) = H(P)/\text{homotheties} .$$

The *Gauss image* of P is the set of outward unit normals to the support planes of P . It defines a convex geodesic cellulation of the sphere \mathbb{S}^2 dual to the one of P . The elements of $\tilde{H}(P)$ are convex polytopes having the same Gauss image than P . In particular:

- elements of $\tilde{H}(P)$ are determined by the (signed) distance of their faces to the origin, called support numbers;
- elements of $\tilde{H}(P)$ all have the same dihedral angles at the corresponding edges and the same face angles at the corresponding vertices;
- in particular, they have the same singular curvature for the induced metric at the corresponding vertices. (These curvatures are positive, see Figure 5.)

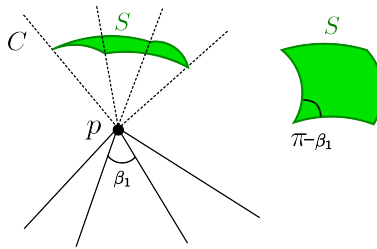


Figure 5: The Gauss image of a vertex p of a convex polyhedron P is the intersection of a cone C , whose edges are orthogonal to the faces of P at p , with a unit sphere centered at p . This gives a spherical convex polygon S , whose interior angles are the interior dihedral angles of C , which correspond to π minus the face angles of P . By the Gauss–Bonnet formula applied to the spherical polygon S , the cone angle around p (the sum of the face angles) is equal to $2\pi - \text{area}(S) < 2\pi$.

As for polygons, we will identify elements of $\tilde{H}(P)$ with their support vectors, i.e. with an element of \mathbb{R}^f .

Parallel faces of elements of $\tilde{H}(P)$ will be considered as convex polygons in a same plane. The identification is through the orthogonal projection onto the linear parallel plane. In this plane, the support numbers of the face are given by the support numbers of the convex polytope, see Figure 6.

From a recursive use of the formula of Figure 3, we get the following fundamental fact the edges length of the elements of $\tilde{H}(P)$ extends to linear forms on \mathbb{R}^f .

As a vector of \mathbb{R}^f is an element of $\tilde{H}(P)$ if and only if it has positive edge length, one obtains that $\tilde{H}(P)$ is a convex polyhedral cone in \mathbb{R}^f . A polytope is *simple* if it has exactly three edges at each vertex. $\tilde{H}(P)$ is open in \mathbb{R}^f if and only if P is simple.

The volume (Lebesgue measure) of P is computed as

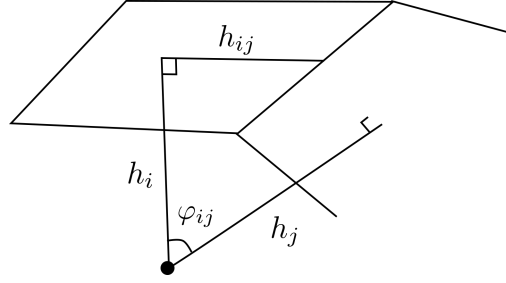


Figure 6: From Figure 3, $h_{ij} = \frac{1}{\sin \varphi_{ij}}(h_j - h_i \cos \varphi_{ij})$.

$$\text{vol}(P) = \frac{1}{3} \sum_{i=1}^m h_i(P) \text{area}(F_i(P))$$

where $F_i(P)$ is the i th face of P , and its area is computed in the plane containing the face.

On \mathbb{R}^f , it extends as a homogenous form of degree 3

$$\text{vol}_P(h) = \frac{1}{3} \sum_{i=1}^m h_i \text{area}_{F_i}(h(i))$$

where $h(i)$ is vector whose entires are given by h_{ij} , varying j , see Figure 6.

the support vector of the i th face seen as a convex polygon. The mixed volume is the symmetric 3-linear form

$$\text{vol}_P(h, k, p) = \frac{1}{3} \sum h_i \text{area}_{F_i}(k(i), p(i)) .$$

Theorem 3.1 (Alexandrov–Fenchel theorem). *Let P be a simple convex polytope. For $h, k, p \in \tilde{H}(P)$,*

$$\text{vol}_P(h, k, p)^2 \geq \text{vol}_P(h, h, p) \text{vol}_P(k, k, p) ,$$

and equality occurs if and only if there exists $x \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$ with $h = h_x + \lambda k$.

The area of P is

$$\text{area}^+(P) = \sum_{i=1}^m \text{area}(F_i(P))$$

it defines the quadratic form

$$\text{area}_P^+(h) = \sum \text{area}_{F_i}(h(i))$$

on \mathbb{R}^f and the symmetric bilinear form

$$\text{area}_P^+(h, k) = \sum \text{area}_{F_i}(h(i), k(i)) .$$

Note that

$$\text{area}_P^+(h) = 3\text{vol}_P(1, h, h), \text{area}_P^+(h, k) = 3\text{vol}_P(1, h, k) .$$

Let us suppose that P is simple and circumscribed, i.e. that the vector $1 = (1, \dots, 1)$ of \mathbb{R}^f belongs to $\tilde{H}(P)$. Then Theorem 3.1 gives the following.

Corollary 3.2. *Let P be a simple circumscribed convex polytope. Then for $h, k \in \tilde{H}(P)$,*

$$\text{area}_P^+(h, k)^2 \geq \text{area}_P^+(h) \text{area}_P^+(k)$$

and equality occurs if and only there exists $x \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$ with $h = h_x + \lambda k$.

Exactly as in the case of polygons in Section 2, we deduce from the result above that area_P^+ has signature $(1, 3, f - 4)$. To remove the assumptions on P , we will need a more

general results on convex bodies. More precisely, Theorem 4.5 (see next section) immediately gives the following statement.

Theorem 3.3. *Let P be a convex polytope. Then for $h, k \in \tilde{H}(P)$,*

$$\text{area}_P^+(h, k)^2 \geq \text{area}_P^+(h) \text{area}_P^+(k)$$

and equality occurs if and only there exists $x \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$ with $h = h_x + \lambda k$.

Corollary 3.4. *Let P be a convex polytope. Then $H_1(P)$ is a convex hyperbolic polyhedron of dimension $\leq f - 4$, which embeds isometrically into $\mathcal{M}(\alpha^+)$.*

Proof. Considering an unfolding of a polytope in the plane, the argument is formally the same as for Corollary 2.4. \square

Remark 3.5. Given a convex polytope P , it may exist a convex polytope P' with faces parallel to the ones of P , but with different combinatorics. Sometimes, as for polygons, $H_1(P)$ can be isometrically glued to $H_1(P')$ along a codimension 1 face. But, conversely to the polygon case, they would embed into different spaces of flat metrics. This is because even if P and P' have the same face angles, they have different cone angles, see Figure 7. Note also that if $Q \in \tilde{H}(P')$, then the support vector of $P + Q$ is not $h_P + h_Q$, see [FI17] for more details.

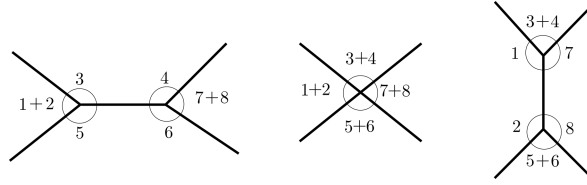


Figure 7: Different combinatorics with same face angles leads to different cone angles.

Example 3.6. A simpler example is when P is a cylinder with a convex polygon P_0 at its base and bottom. Then $H_1(P)$ is a hyperbolic pyramid, with $H_1(P_0)$ as a base, and an ideal point as top. See [FI17] for more details about this examples, and other elementary examples.

4 Area form of convex bodies

Let \mathcal{K}_0^3 be space of convex bodies of \mathbb{R}^3 , i.e. convex compact sets with non-empty interior. The volume $\text{vol}(K)$ of $K \in \mathcal{K}_0^3$ is the Lebesgue measure of K . The *mixed volume* is

$$6\text{vol}(K_1, K_2, K_3) = \text{vol}(K_1 + K_2 + K_3) + \text{vol}(K_1) + \text{vol}(K_2) + \text{vol}(K_3) - \text{vol}(K_1 + K_2) - \text{vol}(K_2 + K_3) - \text{vol}(K_1 + K_3) \quad (4)$$

As K_1, K_2, K_3 can be approximated by a sequence of three simple convex polytopes with parallel faces and same combinatorics, and as the volume of convex sets is continuous, a limit argument and Theorem 3.1 gives the following.

Theorem 4.1 (Alexandrov–Fenchel inequality). *For $K_1, K_2, K_3 \in \mathcal{K}_0^3$,*

$$\text{vol}(K_1, K_2, K_3)^2 \geq \text{vol}(K_1, K_1, K_3) \text{vol}(K_2, K_2, K_3) .$$

The complete description of the equality case is still unknown. However we will need only a particular case.

Let us identify $K \in \mathcal{K}_0^3$ with its support function

$$h_K(v) = \sup_K \langle v, x \rangle$$

defined on \mathbb{S}^2 . Once K_2, K_3 are fixed, one may extend the definition of mixed-covolume for difference of support functions:

$$\text{vol}(h_{K_1} - h_{K_4}, K_2, K_3) := \text{vol}(K_1, K_2, K_3) - \text{vol}(K_4, K_2, K_3).$$

It is easy to see that any C^2 function is the difference of two C^2 support functions. By density, we can extend the definition of $\text{vol}(\cdot, K_2, K_3)$ to $C^0(\mathbb{S}^2)$. By the Riesz representation theorem, there exists a Radon measure $\mu(K_2, K_3)$ on \mathbb{S}^2 such that

$$\text{vol}(h, K_2, K_3) = \int_{\mathbb{S}^2} h \, d\mu(K_2, K_3)$$

In particular, $\mu(K, K)$ is the *area measure* of K , and

$$\text{vol}(K) = \int_{\mathbb{S}^2} h_K \, d\mu(K, K)$$

and

$$\text{area}^+(K) = \int_{\mathbb{S}^2} d\mu(K, K) = \text{vol}(K, K, \mathbf{B})$$

where \mathbf{B} is the closed unit ball, whose support function is the constant 1 function on the sphere. The mixed-area is then

$$\text{area}^+(K_1, K_2) = \text{vol}(K_1, K_2, \mathbf{B})$$

and Alexandrov–Fenchel inequality gives

$$\text{area}^+(K_1, K_2)^+ \geq \text{area}^+(K_1) \text{area}^+(K_2) .$$

But by symmetry of the mixed volume,

$$\text{area}^+(K) = \int_{\mathbb{S}^2} h_K \, d\mu(K, \mathbf{B}) .$$

The classical Christoffel problem asks for existence and uniqueness of a convex body K such that, for a prescribed measure μ , $\mu = \mu(K, \mathbf{B})$. The uniqueness result is as follows. It is a simple consequence of properties of the Laplacian on the sphere, see Theorem 8.3.6 in [Sch14].

Theorem 4.2. *We have $\mu(K_1, \mathbf{B}) = \mu(K_2, \mathbf{B})$ if and only if there exists $x \in \mathbb{R}^3$ such that $K_1 = \{x\} + K_2$.*

Remark 4.3. If P is a convex polytope, $\mu(P, \mathbf{B})$ is the measure on \mathbb{S}^2 supported on the images of the edges of P by the Gauss map, weighted by the lengths of the corresponding edges of P .

Let us combine Theorem 4.2 with the following algebraic lemma. We copied its proof from Lemma 7.4.1 in [Sch14], as we will need it in a future section.

Lemma 4.4. *Let $K_1, K_3 \in \mathcal{K}_0^3$. If*

$$\text{area}^+(K_1, K_3)^2 = \text{area}^+(K_1) \text{area}^+(K_3)$$

then for any $K_2 \in \mathcal{K}_0^3$,

$$\text{area}^+(K_1) \text{area}^+(K_2, K_3) - \text{area}^+(K_1, K_2) \text{area}^+(K_1, K_3) = 0 .$$

Proof. Let $\lambda \geq 0$. On the one hand, by the alexandrov–Fenchel inequality,

$$\text{area}^+(K_2 + \lambda K_3, K_1)^2 - \text{area}^+(K_2 + \lambda K_3, K_2 + \lambda K_3) \text{area}^+(K_1) \geq 0$$

developing an reordering we obtain

$$A\lambda^2 - 2\lambda B + C \geq 0 \tag{5}$$

with

$$A = \text{area}^+(K_1, K_3)^2 - \text{area}^+(K_1) \text{area}^+(K_3)$$

$$B = \text{area}^+(K_2, K_3) \text{area}^+(K_1) - \text{area}^+(K_1, K_2) \text{area}^+(K_1, K_3)$$

$$C = \text{area}^+(K_1, K_2)^2 - \text{area}^+(K_1) \text{area}^+(K_2) .$$

On the other hand, for $t > 0$, Alexandrov–Fenchel inequality gives

$$\text{area}^+(K_2 + t\lambda K_1, K_3 + tK_1)^2 - \text{area}^+(K_2 + t\lambda K_1)\text{area}^+(K_3 + tK_1) \geq 0 .$$

Developing, diving by t^2 and letting $t \rightarrow \infty$, reordering, we obtain

$$A\lambda^2 + 2\lambda B + C \geq 0 . \quad (6)$$

From (5) and (6), for any x , $Ax^2 + 2xB + C$ has constant sign, hence $B^2 - AC \leq 0$. The hypothesis of the lemma says that $A = 0$, that implies $B = 0$. \square

So if

$$\text{area}^+(K_1, K_3)^2 = \text{area}^+(K_1)\text{area}^+(K_3)$$

then, denoting by α the positive number $\text{area}^+(K_1, K_3)/\text{area}^+(K_1)$, for any $K_2 \in \mathcal{K}_0^3$,

$$\text{area}^+(K_2, K_3) = \text{area}^+(K_2, \alpha K_1) .$$

As K_2 is arbitrary, by linearity and density, we obtain

$$\mu(K_3, B) = \mu(\alpha K_1, B)$$

and by Theorem 4.2 we obtain the description of the equality case of Theorem 4.1.

Theorem 4.5. *For $K_1, K_2 \in \mathcal{K}_0^3$,*

$$\text{area}^+(K_1, K_2)^2 \geq \text{area}^+(K_1)\text{area}^+(K_2)$$

with equality if and only if there is $\alpha > 0$ and $x \in \mathbb{R}^3$ such that $K_1 = \{x\} + \alpha K_2$.

This is a reformulation of a theorem of Favard and Kubota, see Theorem 7.6.2 [Sch14].

The argument above has the following heuristic meaning. The area measure is the gradient of the volume, in the sense that $\int_{\mathbb{S}^2} h \, d\mu(K, K)$ is the directional derivative of the volume at K in the direction h , for $h \in C^0(\mathbb{S}^2)$ [Car04]. In the polyhedral case, this is expressed by the fact that $\partial_i \text{vol}(P) = \text{area}(F_i(P))$.

Also, in the same idea, one may consider $\mu(K, \mathbf{B})$ as the derivative $\mu(K, K)$. In the polyhedral case, if $i \neq j$, $\partial_{ij} \text{vol}(P) = \frac{1}{\sin \phi_{ij}} l_{ij}$, where l_{ij} is the length of the edge between the i th face and the j th face (considered as 0 if the faces do not meet along an edge). Alexandrov–Fenchel inequality says that the map

$$(K_1, K_2) \mapsto \text{area}^+(K_1, K_2)^2 - \text{area}^+(K_1)\text{area}^+(K_2)$$

is non-positive. Equality says that it has a critical point, and writing down that the partial derivatives are zero leads to the fact that $\mu(K_1, \mathbf{B})$ and $\mu(K_2, \mathbf{B})$ are proportional.

5 Fuchsian convex polyhedra

Minkowski space $\mathbb{R}^{1,2}$ is \mathbb{R}^3 endowed with the symmetric bilinear form

$$\langle x, y \rangle_{1,2} = x_1 y_1 + x_2 y_2 - x_3 y_3 .$$

Let us denote, for $r > 0$

$$\mathbb{H}_r^2 = \{x | \langle x, x \rangle_{1,2} = -r^2, x_3 > 0\} .$$

The \mathbb{H}_r^2 , $r > 0$, form a foliation of the future cone of the origin $I^+(0) = \{x | \langle x, x \rangle_{1,2} < 0\}$. Note that $\mathbb{H}_1^2 := \mathbb{H}^2$ endowed with the metric induced by the ambient one is a model of the hyperbolic plane.

A plane F in $\mathbb{R}^{1,2}$ is *space-like* if the restriction of $\langle \cdot, \cdot \rangle_{1,2}$ is positive definite. Its *future side* is the half-space bounded by F which contains the direction $(0, 0, 1)$.

Let F_1, \dots, F_n be space-like planes, F_i tangent to $\mathbb{H}_{r_i}^2$ and let Γ be a group of linear isometries of $\mathbb{R}^{1,2}$, such that \mathbb{H}^2/Γ is a compact (oriented) hyperbolic surface.

A Γ *convex polyhedron* P is the intersection of the future side of the orbits of the F_i for Γ . One can see that there exists r, r' such that $\mathbb{H}_r^2 \subset P \subset \mathbb{H}_{r'}^2$, and that the faces are compact convex polygons.

The induced metric on P is flat with conical singularities. As the image of the Gauss map of a vertex is a convex polygon of the hyperbolic space, the singular curvatures at the

vertices are negative (the argument is similar to the one of Figure 5.) In particular, the induced metric on $\partial P/\Gamma$ is a flat metric with cone singularities of negative curvature.

Let us denote by

$$H^\Gamma(P) = \{\Gamma \text{ convex polyhedra with parallel faces and same combinatorics than } P\}$$

As in the case of convex polytopes, one can check that Γ convex polyhedra are determined by a finite numbers of support numbers h_1, \dots, h_m and that $H^\Gamma(P)$ can be identified with a convex polyhedral cone in \mathbb{R}^f . This cone has non empty interior if and only if P is simple.

Similarly to Figure 3 and Figure 6, a support number of a face is computed as ([Fil14, Lemma 2.2]):

$$h_{ij} = -\frac{h(j) - h(i) \cosh \varphi_{ij}}{\sinh \varphi_{ij}}, \quad (7)$$

in particular, edges lengths extend as linear forms of the support vectors,

Note that as Γ is a group of linear isometries, if P_1, P_2 are Γ convex polyhedra, then for $\lambda > 0$, $P_1 + \lambda P_2$ is a Γ convex polyhedron. The *covolume* of P is the volume (given by the Lebesgue measure of \mathbb{R}^3) of $(I^+(0) \setminus P) / \Gamma$. Note that

$$(I^+(0) \setminus P_1) + (I^+(0) \setminus P_2) \neq I^+(0) \setminus (P_1 + P_2) .$$

The covolume may be computed as

$$\text{covol}(P) = \frac{1}{3} \sum h_i \text{area}(F_i)$$

where $\text{area}(F_i)$ is the area of the i th face for the induced metric on the space-like plane containing F_i . Note that, conversly to the covolume, support vector and face area depend on the signature of the bilinear form $\langle \cdot, \cdot \rangle_{1,2}$, see Remark 6.5.

On \mathbb{R}^f , the covolume defines a degree 3 homogenous polynomial

$$\text{covol}_P(h) = \frac{1}{3} \sum_{i=1}^m h_i \text{area}_{F_i}(h(i))$$

where $h(i)$ is the support vector of the i th face seen as a convex polygon. The *mixed covolume* is the symmetric 3-linear form

$$\text{covol}_P(h, k, p) = \frac{1}{3} \sum h_i \text{area}_{F_i}(k(i), p(i)) .$$

Theorem 5.1. *For $h \in H^\Gamma(P)$, $\text{covol}_P(\cdot, \cdot, h)$ is positive definite.*

Proof. By 3-linearity, $D_h^2 \text{covol}_P(k, p) = 3 \text{covol}_P(h, k, p)$, hence it suffices to prove that the Hessian of covol_P at h is positive definite. But also one can check that

$$D_h^2 \text{covol}_P(k, p) = \sum_{i=1}^m p_i D_h \text{area}_{F_i}(k)$$

hence it suffices to study the Jacobian matrix of the map from \mathbb{R}^m to \mathbb{R}^m which associated to h the area of the faces. A straightforward computation gives the result: from (7),

$$\frac{\partial h_{ij}}{\partial h_j} = -\frac{1}{\sinh \varphi_{ij}}, \quad (8)$$

$$\frac{\partial h_{ij}}{\partial h_i} = \frac{\cosh \varphi_{ij}}{\sinh \varphi_{ij}}. \quad (9)$$

If $h_i = h_j$ and if the quadrilateral is deformed under this condition, then

$$\frac{\partial h_{ij}}{\partial h_i} = \frac{\cosh \varphi_{ij} - 1}{\sinh \varphi_{ij}} .$$

Also, recall this property of Euclidean polygons:

$$\frac{\partial A(F_i)}{\partial h_{ik}} = l_{ik}, \quad (10)$$

where l_{ik} is the length of the edge between the face supported by h_i and the one supported by h_k .

We denote by $E_i^j \subset \Gamma\mathcal{I}$ is the set of indices $k \in \Gamma j$ such that F_k is adjacent to F_i along an edge. If $j \in \mathcal{I} \setminus \{i\}$ we get

$$\frac{\partial A(F_i)}{\partial h_j} = \sum_{k \in E_i^j} \frac{\partial A(F_i)}{\partial h_{ik}} \frac{\partial h_{ik}}{\partial h_j}.$$

From (8) and (10) it follows that

$$\frac{\partial A(F_i)}{\partial h_j} = - \sum_{k \in E_i^j} \frac{l_{ik}}{\sinh \varphi_{ik}}. \quad (11)$$

For the diagonal terms:

$$\begin{aligned} \frac{\partial A(F_i)}{\partial h_i} &= \sum_{j \in \mathcal{I} \setminus \{i\}} \sum_{k \in E_i^j} \frac{\partial A(F_i)}{\partial h_{ik}} \frac{\partial h_{ik}}{\partial h_i} + \sum_{k \in E_i^i} \frac{\partial A(F_i)}{\partial h_{ik}} \frac{\partial h_{ik}}{\partial h_i} \\ &\stackrel{(10,9,5)}{=} \sum_{j \in \mathcal{I} \setminus \{i\}} \sum_{k \in E_i^j} \cosh \varphi_{ik} \frac{l_{ik}}{\sinh \varphi_{ik}} + \sum_{k \in E_i^i} l_{ik} \frac{\cosh \varphi_{ik} - 1}{\sinh \varphi_{ik}}. \end{aligned} \quad (12)$$

As $\cosh x > 1$ for $x \neq 0$, (12) and (11) lead to

$$\frac{\partial A(F_i)}{\partial h_i} > \sum_{j \in \mathcal{I} \setminus \{i\}} \left| \frac{\partial A(F_i)}{\partial h_j} \right| > 0$$

that means that the Jacobian is symmetric, strictly diagonally dominant with positive diagonal entries, hence positive definite. \square

The *area* of P is

$$\text{area}^-(P) = \sum_{i=1}^m \text{area}(F_i(P))$$

which defines the following quadratic form on \mathbb{R}^f :

$$\text{area}_P^-(h) = \sum \text{area}_{F_i}(h(i)).$$

Let us suppose that P is simple and circumscribed, i.e. that the vector $1 = (1, \dots, 1)$ of \mathbb{R}^f belongs to $H^\Gamma(P)$. Then Theorem 5.1 with $h = (1, \dots, 1)$ gives the following.

Corollary 5.2. *Let P be a simple circumscribed convex Γ polytope. Then area_P^- is positive definite.*

Let $H_1^\Gamma(P)$ be the quotient of $H^\Gamma(P)$ by homotheties. Hence, if P is simple and circumscribed, $H_1^\Gamma(P)$ is a convex spherical polyhedron (with non empty interior) of \mathbb{H}^f . To remove the assumptions on P , we will need a more general results on Γ convex sets. Namely, Theorem 6.4 in the next section implies the following result.

Theorem 5.3. *Let P be a Γ convex polytope. Then area_P^- is positive definite.*

Similarly to the convex body case, we deduce the following.

Corollary 5.4. *$H_1^\Gamma(P)$ is a convex spherical polyhedron of dimension $\leq f$ which embeds isometrically in a Veech leaf.*

Remark 5.5. There is a similar construction in the Minkowski plane. Formally in the same way than the case of convex compact polygons (see Remark 2.5), one obtains spherical orthoschemes [Fil14]. But they are not related with spaces of flat metrics.

6 Area form on Fuchsian convex bodies

Let \mathcal{K}_Γ^3 be space of Γ convex sets of \mathbb{R}^3 , i.e. convex sets contained in $I^+(0)$ and invariant under the action of Γ (it can be checked that such a convex set has only space-like support planes).

The covolume is defined as in the polyhedral case. The *mixed covolume* is defined formally as the mixed volume in (4).

As K_1, K_2, K_3 can be approximated by a sequence of three simple Γ convex polyhedra with parallel faces and same combinatorics, Theorem 5.1, Cauchy–Schwarz inequality and a limit argument give the following.

Theorem 6.1 (Reversed Alexandrov–Fenchel inequality). *For $K_1, K_2, K_3 \in \mathcal{K}_\Gamma^3$,*

$$\text{covol}(K_1, K_2, K_3)^2 \leq \text{covol}(K_1, K_1, K_3) \text{covol}(K_2, K_2, K_3) .$$

Let us identify $K \in \mathcal{K}_\Gamma^3$ with its support function

$$h_K(v) = \sup_K \langle v, x \rangle_{1,2}$$

which is Γ invariant. We consider h_K as a function on the compact surface \mathbb{H}^2/Γ . One can also prove that there exists a Radon measure $\mu(K_2, K_3)$ on \mathbb{H}^2/Γ such that [FV16, Remark 3.15]

$$\text{covol}(h, K_2, K_3) = \int_{\mathbb{H}^2/\Gamma} h \, d\mu(K_2, K_3) .$$

In particular, $\mu(K, K)$ is the area measure of K , and

$$\text{covol}(K) = \int_{\mathbb{H}^2/\Gamma} h_K \, d\mu(K)$$

and

$$\text{area}^-(K) = \int_{\mathbb{H}^2/\Gamma} d\mu(K, K) = \text{vol}(K, K, \mathbf{H})$$

where \mathbf{H} is the Γ convex set bounded by \mathbb{H}^2 .

Reversed Alexandrov–Fenchel inequality (Theorem 6.1) then reads as follows: for $K_1, K_2 \in \mathcal{K}_\Gamma^3$,

$$\text{area}^-(K_1, K_2)^2 \leq \text{area}^-(K_1) \text{area}^-(K_2) .$$

But by symmetry of the mixed volume,

$$\text{area}^-(K) = \int_{\mathbb{H}^2/\Gamma} h_K \, d\mu(K, \mathbf{H}) .$$

The Γ -Christoffel problem asks for existence and uniqueness of a Γ -convex body K such that, for a prescribed measure μ on \mathbb{H}^2/Γ , $\mu = \mu(K, \mathbf{H})$. The uniqueness result is as follows. It is a simple consequence of properties of Laplacian on a compact hyperbolic manifold.

Theorem 6.2 ([FV16]). *We have $\mu(K_1, \mathbf{H}) = \mu(K_2, \mathbf{H})$ if and only if $K_1 = K_2$.*

The proof of the following lemma is formally the same than the proof of Lemma 4.4.

Lemma 6.3. *Let $K_1, K_3 \in \mathcal{K}_\Gamma^3$. If*

$$\text{area}^-(K_1, K_3)^2 = \text{area}^-(K_1) \text{area}^-(K_3)$$

then for any $K_2 \in \mathcal{K}_\Gamma^3$,

$$\text{area}^-(K_1) \text{area}^-(K_2, K_3) - \text{area}^-(K_1, K_2) \text{area}^-(K_1, K_3) = 0 .$$

So, exactly as in the Euclidean case, if equality occurs in the reversed Alexandrov–Fenchel inequality, there will exists a positive α such that

$$\mu(K_3, \mathbf{H}) = \mu(\alpha K_1, \mathbf{H})$$

and by Theorem 6.2 and Theorem 6.1 we obtain the following.

Theorem 6.4. *For $K_1, K_2 \in \mathcal{K}_\Gamma^3$,*

$$\text{area}^+(K_1, K_2)^2 \leq \text{area}^+(K_1) \text{area}^+(K_2)$$

with equality if and only if there is $\alpha > 0$ such that $K_1 = \alpha K_2$.

Remark 6.5. There is no need of an ambient Lorentzian metric to state the reversed Alexandrov–Fenchel Theorem 6.1. Actually this is true for the category of *coconvex sets*, and can be deduced from the classical Alexandrov–Fenchel inequality for convex bodies, see [KT14]. Also, it can be proved that the covolume is strictly convex, see Figure 8. So it is natural to think about the following statement.

Question 6.6. Equality occurs in reversed Alexandrov–Fenchel inequality if and only if $K_1 = \lambda K_2$ for some $\lambda > 0$?

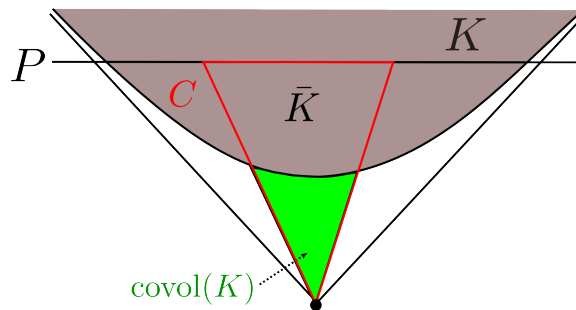


Figure 8: Let C be truncated cone which is the intersection of a (convex) fundamental domain for Γ with a half-space bounded by P . Let \bar{K} be the intersection of C with K . Then $\text{covol}(K) = \text{vol}(C) - \text{vol}(\bar{K})$. \bar{K} is a kind of convex body known as *convex cap*. A simple application of Fubini theorem gives that volume on the set of convex caps with given basis is strictly concave. Hence the covolume is strictly convex. See [BF14] for more details.

7 Acknowledgement

This note is based in remarks 4.3 and 4.4 in [FI17], see also [Fil13, Fil14]. Theorem 3.3 of the present paper is Corollary 2.9 in [FI17]. Theorem 5.1 comes from [Fil13]. Theorem 5.3 of the present paper is new. The subject of [FI17] is to endow different spaces of convex polytopes with parallel faces (in any dimension) with hyperbolic structures using the mixed volume.

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